**Spin in B field**

**Hamiltonian of an electron spin in a magnetic field**

Consider an electron in a magnetic field, **B**. It has spin angular momentum **S**, and we’d like to know how the electron is affected in a magnetic field. For instance, how it would evolve in such a field, and what the steady state energy levels would be. For point of reference we can consider a classical dipole in a magnetic field. A dipole with constant current (must be constant because current/magnetic moment is directly proportional to spin angular momentum, and the magnitude of this is fixed – as saw in previous file), in a magnetic field, would be rotated by it. The magnetic field would exert a torque **τ** = **μ**×**B**, and as the dipole rotates in that field, the field would do work:



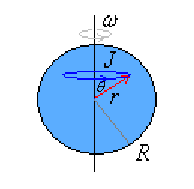
where we might recall the vector product rule **A**·(**B**×**C**) = **B**·(**C**×**A**) = **C**·(**A**×**B**), and how if a vector of constant magnitude, **Q**, rotates by δ**θ**, then it changes by δ**θ**×**Q** (see Classical Mechanics, accelerated reference frames, e.g.). Forming H = PE = -∫dW, we come to:



Parenthetically, should comment that this is only half the picture. Classically, the magnetic field would also do back-emf’ work (see Physics 2 stuff), equal and opposite to the torque work, against the current in the magnetic moment (in accordance with the established fact that classically, the magnetic field can do *no* *net* work). Thus classically, the magnetic field would rotate our dipole *and* reduce its dipole moment at the same time. But as we’ll see, reducing the magnitude of the electron’s dipole moment would amount to reducing the electron’s spin angular momentum. And this cannot happen, as we saw previously that it’s fixed with s = ½ quantum number. So in some sense, the electron has an internal power source maintaining its magnetic moment against external influences. And so it *seems* the magnetic field *does* do work on our electron. But I’d rather chalk this up to the induced electric field coming from our changing B field as the source of any work that is being done on our particle. As discussed in the Thermodynamics folder (Equilibrium Systems), it really doesn’t fit with the rest of physics to say that the magnetic field does work.

One final note: as repeated elsewhere, this H should be taken as simply a kind of rotational kinetic energy for the spin, which only manifests in a magnetic field. It is not a ‘potential’ energy. Can think of it as analogous to the **L**·**B** term (L being orbital angular momentum) that arises when we work out (**p**-e**A**)/2m in the symmetric gauge **A** = -(1/2)**r**×**B**. In further support of this identification, we’ll note that when we do relativistic QM, this energy term comes out naturally from the relativistic KE.

Well if we suppose H = -**μ**·**B**. What remains is to figure out **μ**. For the sake of discussion, suppose it has angular momentum **S**, classically speaking. And suppose that its charge is evenly distributed throughout its entire volume. **J** is the current density at the point **r**.



The current density at a point in the electron’s volume is **J** = ρ­c**v**, where ρc is the charge density at that point (this would just be ρc = q/[4πR3/3]) times the velocity of the differential piece of charge at that point. Now if the electron is rotating with angular velocity ω, and the position vector, **r**, to that differential piece of charge is making an angle θ w/r to the z-axis (the usual polar angle from spherical coordinates), then the velocity of that differential piece of charge would be:



And so the current density at that point would be:



And so then calculating the magnetic dipole moment,



where **s** is a vector which points inward towards the z-axis, parallel to the x-y plane. You can think about that cross product if you want. Anyway, the point is that the integral over the **s** component will give us 0 because points on opposite sides of the electron will cancel out. So only the z-component survives, which should be the case anyway. So we have:



OK, now let’s compute what the spin-angular momentum of the electron would be, classically. It would be (remembering the formula for the moment of inertia of a sphere):



Comparing μ and S, turns out we still have:



In fact, it is a general truth that for classical objects (see EM file – Asymptotic Expansion), with charge evenly distributed throughout their volume, their magnetic dipole moment is directly proportional to their spin angular momentum (i.e. angular momentum about their center of mass) via γ = q/2m. The electron, however, anomalously, does not *quite* obey this relation, as for it we have:



(e is negative for electrons, positive for protons) This ‘anomalous’ g-factor is g ≈ 2.0023. The g-factor is completely inexplicable at the moment. When we get to the Dirac equation, it will predict that g ought to be 2, which is almost correct. Quantum Field Theory is necessary for the rest of it. So we end up with:



**Eigenvectors/values of spin in magnetic field**

Alright, so given that, we can write the energy of a fixed (can’t move) particle in a magnetic field as:



and assuming B points in the z-direction as before, this reduces to:



The eigenfunctions of this Hamiltonian are of course the eigenfunctions of the Sz operator, χ+= (1 0) (the spin up state where the magnetic moment and magnetic field are anti-aligned) and χ- = (0 1) (the spin down state where the magnetic moment and magnetic field are aligned). So we have:



where we define the Bohr magneton,



Note that the energy matches the frequency (γB) of rotation of a classical spin. This even though the quantum spin has the anomalous g-factor. Interesting!

(T means Tesla) This quantity has units of magnetization, as [Mag] = CL2/T, and [μB] = [ℏ][e]/[m] = ML2/T·C/M = CL2/T (T means time now).